

# SELF-ADJOINTNESS OF THE SEMI-RELATIVISTIC PAULI-FIERZ HAMILTONIAN

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## Abstract

The spinless semi-relativistic Pauli-Fierz Hamiltonian

$$H = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2} + V \otimes \mathbb{1} + \mathbb{1} \otimes H_f,$$

in quantum electrodynamics is considered. Here  $p$  denotes a momentum operator,  $A$  a quantized radiation field,  $M \geq 0$ ,  $H_f$  the free hamiltonian of a Boson Fock space and  $V$  an external potential. The self-adjointness and essential self-adjointness of  $H$  are shown. It is emphasized that it includes the case of  $M = 0$ . Furthermore, the self-adjointness and the essential self-adjointness of the semi-relativistic Pauli-Fierz model with a fixed total momentum  $P \in \mathbb{R}^d$ :

$$H(P) = \sqrt{(P - P_f - A(0))^2 + M^2} + H_f, \quad M \geq 0,$$

is also proven for arbitrary  $P$ .

## 1 Introduction

### 1.1 Fundamental facts

In this paper we are concerned with the self-adjointness of the so-called semi-relativistic Pauli-Fierz (SRPF) Hamiltonian  $H$  in quantum electrodynamics. Essential self-adjointness

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of  $H$  is shown in [Hir14, Theorem 4.5] by a path measure approach under some conditions. We furthermore show its self-adjointness under weaker conditions in this paper. Our result is independent of coupling constants. In this sense the result is non-perturbative.

Let  $\mathcal{H}$  be a Hilbert space over  $\mathbb{C}$  and  $h$  be a symmetric operator with the domain  $D_0$ . In general  $h$  has the infinite number of self-adjoint extensions. Let  $h_0$  be one self-adjoint extension, which defines the Schrödinger equation

$$i\frac{\partial}{\partial t}\Phi_t = h_0\Phi_t \quad (1.1)$$

with the initial condition  $\Phi_0 = \Phi \in \mathcal{H}$ . Then the self-adjointness of  $h_0$  ensures the uniqueness of the solution to (1.1) and it is given by  $\Phi_t = e^{-ith_0}\Phi$ . The time-evolution of a physical system governed by the Schrödinger equation (1.1) is different according to which self-adjoint extension is chosen. Hence it is important to find a core of  $h$  or a domain on which  $h$  is self-adjoint in order to determine the unique time-evolution of the physical system.

A semi-relativistic Schrödinger operator with nonnegative rest mass  $M \geq 0$  is defined as a self-adjoint operator in  $L^2(\mathbb{R}^d)$ , which is given by

$$H_p = \sqrt{p^2 + M^2} + V. \quad (1.2)$$

Here  $p = (-i\partial_{x_1}, \dots, -i\partial_{x_d})$  denotes the momentum operator and  $V : \mathbb{R}^d \rightarrow \mathbb{R}$  is an external potential. The SRPF model is defined by  $H_p$  coupled to a quantized radiation field  $A$ . Let  $\mathcal{F} = \bigoplus_{n=0}^{\infty} \mathcal{F}_n(W) = \bigoplus_{n=0}^{\infty} \bigotimes_s^n W$  be the Boson Fock space over Hilbert space  $W = \bigoplus^{d-1} L^2(\mathbb{R}^d)$ ,  $d \geq 3$ . Although the physically reasonable choice of the spatial dimension is  $d = 3$ , we generalize it. Let  $\omega : \mathbb{R}^d \rightarrow \mathbb{R}$  be a dispersion relation. We introduce assumptions on the dispersion relation.

**Assumption 1.1**  $\omega(k) \geq 0$  a.e.  $k \in \mathbb{R}^d$ .

Physically reasonable choice of dispersion relation is  $\omega(k) = |k|$  or  $\omega(k) = \sqrt{|k|^2 + \nu^2}$  with some  $\nu > 0$ . In [HH13] the dispersion relation such that  $\omega \in C^1(\mathbb{R}^d; \mathbb{R})$ ,  $\nabla\omega \in L^\infty(\mathbb{R}^d)$ ,  $\inf_{k \in \mathbb{R}^d} \omega(k) \geq m$  with some  $m > 0$  and  $\lim_{|k| \rightarrow \infty} \omega(k) = \infty$  is treated. The free field Hamiltonian  $H_f$  of the Boson Fock space is given by the second quantization of the multiplication operator by  $\omega$  on  $W$ , i.e.,  $H_f = d\Gamma(\omega)$ . The SRPF Hamiltonian is defined by the minimal coupling of a quantized radiation field  $A$  to

$$H_0 = H_p \otimes \mathbb{1} + \mathbb{1} \otimes H_f. \quad (1.3)$$

$H_0$  is self-adjoint on  $D(H_p \otimes \mathbb{1}) \cap D(\mathbb{1} \otimes H_f)$ . The creation operator and the annihilation operator in  $\mathcal{F}$  are denoted by  $a^\dagger(f)$  and  $a(f)$ ,  $f \in W$ , respectively. They

are linear in  $f$  and satisfy canonical commutation relations:  $[a(f), a^\dagger(g)] = (\bar{f}, g)_W$  and  $[a(f), a(g)] = 0 = [a^\dagger(f), a^\dagger(g)]$ . Here and in what follows the scalar product  $(f, g)_\mathcal{H}$  on a Hilbert space  $\mathcal{H}$  is linear in  $g$  and anti-linear in  $f$ . We formally write as  $a^{\#r}(f) = \int a^{\#r}(k)f(k)dk$  for  $a^\#(F)$  with  $F = \bigoplus_{s=1}^{d-1} \delta_{sr}f$  and

$$H_f = \sum_{r=1}^{d-1} \int \omega(k) a^{\dagger r}(k) a^r(k) dk.$$

Let  $e^r(k) = (e_1^r(k), \dots, e_d^r(k))$  be  $d$ -dimensional polarization vectors, i.e.,  $e^r(k) \cdot e^s(k) = \delta_{rs}$  and  $k \cdot e^r(k) = 0$  for  $k \in \mathbb{R}^d \setminus \{0\}$  and  $r = 1, \dots, d-1$ . For each  $x \in \mathbb{R}^d$  a quantized radiation field  $A(x) = (A_1(x), \dots, A_d(x))$  is defined by

$$A_\mu(x) = \frac{1}{\sqrt{2}} \sum_{r=1}^{d-1} \int e_\mu^r(k) \left( \frac{\hat{\varphi}(k)e^{-ik \cdot x}}{\sqrt{\omega(k)}} a^{\dagger r}(k) + \frac{\hat{\varphi}(-k)e^{ik \cdot x}}{\sqrt{\omega(k)}} a^r(k) \right) dk. \quad (1.4)$$

Here  $\hat{\varphi}$  is an ultraviolet cutoff function, for which we introduce assumptions below.

**Assumption 1.2**  $\hat{\varphi}/\sqrt{\omega}, \omega\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^d)$  and  $\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}$ .

Note that  $\sqrt{\omega}\hat{\varphi} \in L^2(\mathbb{R}^d)$  follows from Assumption 1.2. We fix  $\hat{\varphi}$  and  $\omega$  satisfying Assumptions 1.1 and 1.2 throughout this paper. Then  $\hat{\varphi}(k) = \overline{\hat{\varphi}(-k)}$  implies that  $A_\mu(x)$  is essentially self-adjoint for each  $x$ . We denote the self-adjoint extension by the same symbol  $A_\mu(x)$ . We identify  $\mathcal{H}$  with  $\int_{\mathbb{R}^d}^\oplus \mathcal{F} dx$ , and under this identification we define the self-adjoint operator  $A_\mu$  in  $\mathcal{H}$  by

$$A_\mu = \int_{\mathbb{R}^d}^\oplus A_\mu(x) dx.$$

Set  $A = (A_1, \dots, A_d)$ . Let  $N = d\Gamma(\mathbb{1})$  be the number operator on  $\mathcal{F}$  and  $C^\infty(\mathbb{1} \otimes N) = \bigcap_{n=1}^\infty D(\mathbb{1} \otimes N^n)$ . Let

$$\sum_{\mu=1}^d (p_\mu \otimes \mathbb{1} - A_\mu)^2 = (p \otimes \mathbb{1} - A)^2. \quad (1.5)$$

**Lemma 1.3**  $D(p^2 \otimes \mathbb{1}) \cap C^\infty(\mathbb{1} \otimes N) \cap D(\mathbb{1} \otimes H_f)$  is a core of  $(p \otimes \mathbb{1} - A)^2$ .

*Proof:* See Appendix B. ■

The closure of  $(p \otimes \mathbb{1} - A)^2 \upharpoonright_{D(p^2 \otimes \mathbb{1}) \cap C^\infty(\mathbb{1} \otimes N) \cap D(\mathbb{1} \otimes H_f)}$  is denoted by  $(p \otimes \mathbb{1} - A)^2$  for simplicity. Thus  $\sqrt{(p \otimes \mathbb{1} - A)^2 + M^2}$  is defined through the spectral measure of  $(p \otimes \mathbb{1} - A)^2$ . Set

$$T_M = \sqrt{(p \otimes \mathbb{1} - A)^2 + M^2}. \quad (1.6)$$

**Proposition 1.4** [Hir14, Lemma 3.12, Theorem 4.5] *Let  $M > 0$ . Then (1) and (2) follow.*

- (1) *Let  $V = 0$ . Then  $H$  is essentially self-adjoint on  $\mathcal{D}$ .*
- (2) *Suppose that  $V$  is relatively bounded (resp. form bounded) with respect to  $\sqrt{p^2 + M^2}$  with a relative bound  $a$ . Then  $V$  is also relatively bounded (resp. form bounded) with respect to  $T_M + H_f$  with a relative bound smaller than  $a$ .*

## 1.2 Potential classes and definition of SRPF Hamiltonian

We introduce two classes,  $V_{\text{qf}}$  and  $V_{\text{rel}}$ , of potentials.

**Definition 1.5** ( $V_{\text{qf}}$ )  $V = V_+ - V_- \in V_{\text{qf}}$  if and only if  $V_+ \in L^1_{\text{loc}}(\mathbb{R}^d)$  and  $V_-$  is relatively form bounded with respect to  $\sqrt{p^2 + M^2}$  with a relative bound strictly smaller than one, i.e.,  $D((p^2 + M^2)^{1/4}) \subset D(V_-^{1/2})$  and there exist  $0 \leq a < 1$  and  $b \geq 0$  such that

$$\|V_-^{1/2}f\| \leq a\|(p^2 + M^2)^{1/4}f\| + b\|f\|$$

for all  $f \in D((p^2 + M^2)^{1/4})$ .

( $V_{\text{rel}}$ )  $V \in V_{\text{rel}}$  if and only if  $V$  is relatively bounded with respect to  $\sqrt{p^2 + M^2}$  with a relative bound strictly smaller than one, i.e.,  $D(\sqrt{p^2 + M^2}) \subset D(V)$  and there exist  $0 \leq a < 1$  and  $b \geq 0$  such that

$$\|Vf\| \leq a\|\sqrt{p^2 + M^2}f\| + b\|f\|$$

for all  $f \in D(\sqrt{p^2 + M^2})$ .

It can be shown that  $V_{\text{rel}} \subset V_{\text{qf}}$ . By Proposition 1.4 we can define the SRPF Hamiltonian as a self-adjoint operator through quadratic form sums. Let  $V \in V_{\text{qf}}$ . We define the quadratic form by

$$q : (F, G) \mapsto (T_M^{1/2}F, T_M^{1/2}G) + (H_f^{1/2}F, H_f^{1/2}G) + (V_+^{1/2}F, V_+^{1/2}G) - (V_-^{1/2}F, V_-^{1/2}G) \quad (1.7)$$

with the form domain

$$Q(q) = D(T_M^{1/2}) \cap D(H_f^{1/2}) \cap D(V_+^{1/2}). \quad (1.8)$$

By Proposition 1.4, we note that  $Q(q) = D(T_M^{1/2}) \cap D(H_f^{1/2}) \cap D(V_+^{1/2}) \cap D(V_-^{1/2})$ . It can be checked that  $Q(q)$  is densely defined semi-bounded closed form. Then there exists the unique self-adjoint operator  $H$  associated with the quadratic form  $q$ , i.e.,  $D(|H|^{1/2}) = Q(q)$  and  $q(F, G) = \int_{\sigma(H)} \lambda d(E_\lambda F, G)$ . Here  $E_\lambda$  denotes the spectral measure associated with  $H$ . We write  $H$  as

$$H = T_M + V_+ \otimes \mathbb{1} + V_- \otimes \mathbb{1} + \mathbb{1} \otimes H_f. \quad (1.9)$$

**Definition 1.6** Let  $V \in V_{\text{qf}}$ . Then the SRPF Hamiltonian is defined by (1.9).

We do not write tensor notation  $\otimes$  for notational convenience in what follows. Thus  $H$  can be simply written as  $H = T_M + H_f + V_+ - V_-$ .

### 1.3 Essential self-adjointness of $H$

Let

$$\mathcal{D} = D(|p|) \cap D(V) \cap D(H_f). \quad (1.10)$$

When  $V \in V_{\text{rel}}$ ,  $D(V) \subset D(|p|) \cap D(H_f)$  and it follows that  $\mathcal{D} = D(|p|) \cap D(H_f)$ . We introduce a subclass  $V_{\text{conf}} \subset V_{\text{qf}}$ , which include confining potentials.

**Definition 1.7** ( $V_{\text{conf}}$ )  $V = V_+ - V_- \in V_{\text{conf}}$  if and only if  $V_- = 0$  and  $V_+$  is twice differentiable, and  $\partial_\mu V_+, \partial_\mu^2 V_+ \in L^\infty(\mathbb{R}^d)$  for  $\mu = 1, \dots, d$ , and  $D(V) \subset D(|x|)$ .

When  $V \in V_{\text{conf}}$ ,  $V \in L_{\text{loc}}^2(\mathbb{R}^d)$  and nonnegative. Then  $p^2 + V$  is essentially self-adjoint on  $C_c^\infty(\mathbb{R}^d)$  by Kato's inequality. It is established in [Hir14, Theorem 4.5] that  $H$  with  $M > 0$  is essentially self-adjoint on  $\mathcal{D}$  for  $V \in V_{\text{rel}}$ . We extend this to  $V \in V_{\text{rel}} \cup V_{\text{conf}}$ .

**Proposition 1.8** Let  $V \in V_{\text{rel}} \cup V_{\text{conf}}$  and  $M > 0$ . Then  $H$  is essentially self-adjoint on  $\mathcal{D}$ .

*Proof:* When  $V \in V_{\text{rel}}$ , the proposition follows from (2) of Proposition 1.4 and the Kato-Rellich theorem. The proof of the proposition for  $V \in V_{\text{conf}}$  is a minor modification of [Hir14, Theorem 4.5]. Then we give it in Appendix C.  $\blacksquare$

### 1.4 Main results

The self-adjointness of the Pauli-Fierz Hamiltonian:

$$\frac{1}{2}(p \otimes \mathbb{1} - \alpha A)^2 + V \otimes \mathbb{1} + \mathbb{1} \otimes H_f$$

is proven in [HH08, Hir00, Hir02] for arbitrary values of coupling constant  $\alpha \in \mathbb{R}$  under some condition on  $\hat{\varphi}$  and  $V$ . On the other hand as far as we know there a few work on the self-adjointness of the SRPF Hamiltonian. In [MS09] the self-adjointness of the SRPF Hamiltonian with spin 1/2 and without  $V$ :

$$\gamma \sqrt{(\sigma \cdot (p \otimes \mathbb{1} - \alpha A))^2 + M^2} + \mathbb{1} \otimes H_f$$

is shown for  $d = 3$  but for sufficiently small coupling constant  $\alpha$ , where  $\sigma = (\sigma_1, \sigma_2, \sigma_3)$  denotes  $2 \times 2$  Pauli matrices and  $0 < \gamma \leq 1$  an artificial parameter. In [MS09] the

self-adjointness is proven by a perturbation theory, i.e., operator  $|D| - |D_0|$  is estimated for sufficiently small  $\alpha$ , where  $|D| = \sqrt{(\sigma \cdot (p \otimes \mathbb{1} - \alpha A))^2 + M^2}$  and  $|D_0| = \sqrt{(\sigma \cdot p)^2 + M^2}$ , and the self-adjointness of  $|D| + H_f$  can be reduce to show that of  $|D_0| + H_f$  for sufficiently small  $\alpha$ . This is unfortunately not applicable for arbitrary values of  $\alpha$ . By functional integration however it is proven in [Hir14] that  $H$  is essentially self-adjoint on  $\mathcal{D}$  for  $M > 0$ , which is due to show that  $e^{-tH}\mathcal{D} \subset \mathcal{D}$ .

Then the main purpose of this paper is to show the self-adjointness of  $H$  on  $\mathcal{D}$  for arbitrary values of coupling constants (in this paper  $\alpha$  is absorbed in the prefactor of  $\hat{\varphi}$ ), and not only for  $V \in V_{\text{rel}}$  but also for  $V \in V_{\text{conf}}$ . This can be achieved by proving the nontrivial bound (2.2) mentioned below, which bound implies the closedness of  $H|_{\mathcal{D}}$ . In order to prove (2.2) for  $0 \leq M$  we have to estimate the commutator like  $[\sqrt{(p - A)^2 + M^2}, \cdot]$ . See the proof of Lemma 2.6. In particular the proof for the case of  $M = 0$  is not technically straightforward, and then we used a functional integral method.

We define the dense subset  $\mathcal{H}_{\text{fin}}$ . Let

$$\mathcal{F}_{\text{fin}} = L.H.\{\Omega, a^\dagger(h_1) \cdots a^\dagger(h_n)\Omega | h_j \in C_c^\infty(\mathbb{R}^d), j = 1, \dots, n, n \geq 1\} \quad (1.11)$$

and

$$\mathcal{H}_{\text{fin}} = C_c^\infty(\mathbb{R}^d) \hat{\otimes} \mathcal{F}_{\text{fin}}, \quad (1.12)$$

where  $\hat{\otimes}$  denotes the algebraic tensor product. The main theorem in this paper is to extend Proposition 1.8 as follows.

**Theorem 1.9** *Let  $V \in V_{\text{rel}} \cup V_{\text{conf}}$  and  $M \geq 0$ . Then  $H$  is self-adjoint on  $\mathcal{D}$ , and essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ .*

Note that Theorem 1.9 includes the case of  $M = 0$ .

## 1.5 Literatures and organization

We refer to literatures where the SRPF model is studied. In [GLL01] the existence of the ground state of the SRPF model is suggested and the present work is inspired from this. Then the ground state of the SRPF model is studied in e.g., [GS12, HH13, Hir14, KM13a, KM13b, KM14, KMS11a, KMS11b, MS10, MS09], in particular the case of  $M = 0$  is investigated in [HH13]. Moreover in [FGS01] the asymptotic analysis of the SRPF model is also studied.

This paper is organized as follows.

In Section 2 we show that  $H$  is self-adjoint on  $D(|p|) \cap D(V) \cap D(H_f)$  and essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$  which is defined in (1.12).

In Section 3 we discuss the translation invariant SRPF Hamiltonian which is defined by  $H$  with  $V = 0$ . Then  $H \cong \int_{\mathbb{R}^d}^\oplus H(P) dP$  is obtained and  $H(P)$  is called the SRPF

Hamiltonian with total momentum  $P \in \mathbb{R}^d$ . The self-adjointness of  $H(P)$  on  $D(|P_f|) \cap D(H_f)$ , and essential self-adjointness on  $\mathcal{F}_{\text{fin}}$  defined in (1.11).

## 2 Self-adjointness

In order to prove Theorem 1.9 we need several lemmas.

**Lemma 2.1** *Let  $M \geq 0$ . It follows that  $D(|p|) \cap D(H_f^{1/2}) \subset D(T_M)$ , and for all  $\Psi \in D(|p|) \cap D(H_f^{1/2})$ ,*

$$\|T_M \Psi\| \leq C(\| |p| \Psi \| + \|H_f^{1/2} \Psi\| + \|\Psi\|) \quad (2.1)$$

*with some constant  $C > 0$ . In particular*

$$\|H \Psi\| \leq C(\| |p| \Psi \| + \|H_f \Psi\| + \|V \Psi\| + \|\Psi\|) \quad (2.2)$$

*follows for  $\Psi \in \mathcal{D}$  with some constant  $C > 0$ .*

*Proof:* It follows that  $\|T_M \Psi\|^2 = \sum_{\mu=1}^d \|(p_\mu - A_\mu) \Psi\|^2 + M^2 \|\Psi\|^2$  for  $\Psi \in \mathcal{H}_{\text{fin}}$ . Then (2.1) follows from the well-known bound

$$\|A_\mu \Psi\| \leq C \|(H_f + \mathbb{1})^{1/2} \Psi\|$$

with some constant  $C > 0$  for  $\Psi \in \mathcal{H}_{\text{fin}}$ . Furthermore since  $|p| + H_f^{1/2}$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ , the lemma follows from a limiting argument.  $\blacksquare$

Let  $\mathcal{H}_0 = \{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{H} \mid \Psi^{(n)} = 0 \text{ for all } n \geq n_0 \text{ with some } n_0 \geq 1\}$  and

$$\mathcal{D}_1 = \mathcal{D} \cap \mathcal{H}_0. \quad (2.3)$$

**Lemma 2.2** *Let  $V \in V_{\text{conf}}$  and  $M > 0$ . Then  $\mathcal{D}_1$  is a core of  $H$ .*

*Proof:* Let  $P_n = \mathbb{1}_{[0,n]}(N)$  for  $n \in \mathbb{N}$ . Take an arbitrary  $\Psi \in \mathcal{D}$ . Hence  $P_n \Psi \in \mathcal{D}_1$ . We see that  $P_n \Psi \rightarrow \Psi$  as  $n \rightarrow \infty$ . Since

$$\|H(P_n - P_{n'}) \Psi\| \leq C(\|(P_n - P_{n'}) |p| \Psi\| + \|(P_n - P_{n'}) V \Psi\| + \|(P_n - P_{n'}) H_f \Psi\|), \quad (2.4)$$

we also see that  $\{H P_n \Psi\}_n$  is a Cauchy sequence in  $\mathcal{H}$ . By the closedness of  $H$ ,  $\Psi \in D(H)$  and  $H P_n \Psi \rightarrow H \Psi$ . Thus  $\mathcal{D}_1$  is a core of  $H$ .  $\blacksquare$

Let

$$\mathcal{D}_2 = \{\{\Psi^{(n)}\}_{n=0}^\infty \in \mathcal{D}_1 \mid \Psi^{(n)}(\cdot, \mathbf{k}) \in C_c^\infty(\mathbb{R}^d) \text{ a.e. } \mathbf{k} \in \mathbb{R}^{dn}, n \geq 1\}. \quad (2.5)$$

**Lemma 2.3** *Let  $V \in V_{\text{conf}}$  and  $M > 0$ . Then  $\mathcal{D}_2$  is a core of  $H$ .*

*Proof:* Take an arbitrary  $\Phi \in \mathcal{D}_1$ . Let  $j \in C_c^\infty(\mathbb{R}^d)$  and  $g \in C_c^\infty(\mathbb{R}^d; [0, 1])$  such that  $\int_{\mathbb{R}^d} j(x) dx = 1$  and  $g(x) = 1$  for  $|x| \leq 1$ . For each  $\epsilon > 0$  we set  $j_\epsilon(x) = \epsilon^{-d} j(x/\epsilon)$ ,

$$\Phi_{\epsilon, L}^{(n)}(x, \mathbf{k}) = g(x/L) \int_{\mathbb{R}^d} j_\epsilon(x - y) \Phi^{(n)}(y, \mathbf{k}) dy, \quad (2.6)$$

and  $\Phi_{\epsilon, L} = \{\Phi_{\epsilon, L}^{(n)}\}_{n=0}^\infty$ . We see that  $\Phi_{\epsilon, L} \rightarrow \Phi$ ,  $p_\mu \Phi_{\epsilon, L} \rightarrow p_\mu \Phi$ ,  $V \Phi_{\epsilon, L} \rightarrow V \Phi$  and  $H_f \Phi_{\epsilon, L} \rightarrow H_f \Phi$  strongly as  $\epsilon \downarrow 0$  and  $L \rightarrow \infty$ . Then by inequality (2.2) and the closedness of  $H$ , we see that  $\Phi \in D(H)$  and  $\lim_{L \rightarrow \infty} \lim_{\epsilon \downarrow 0} H \Phi_{\epsilon, L} = H \Phi$  in  $\mathcal{H}$ . Thus the lemma follows. ■

**Lemma 2.4** *Let  $V \in V_{\text{conf}}$  and  $M > 0$ . Let  $\Phi \in \mathcal{D}_2$ . Then it follows that*

$$\|p^2 \Phi\| + \|V \Phi\| + \|H_f \Phi\| \leq C \|(p^2 + V + H_f + \mathbb{1}) \Phi\| \quad (2.7)$$

with some constant  $C > 0$ .

*Proof:* Note that  $\|(p^2 + V) \Phi\|^2 = \|p^2 \Phi\|^2 + 2\text{Re}(p^2 \Phi, V \Phi) + \|V \Phi\|^2$ . Let  $V_\mu = \partial_\mu V$ . Since

$$2\text{Re}(p^2 \Phi, V \Phi) \geq 2 \sum_{\mu} \text{Re}(p_\mu \Phi, [p_\mu, V] \Phi) \geq -2 \sum_{\mu} \|p_\mu \Phi\| \|V_\mu\|_\infty \|\Phi\|,$$

for an arbitrary  $\epsilon > 0$ , we have  $\|(p^2 + V) \Phi\|^2 \geq (1 - \epsilon) \|p^2 \Phi\|^2 + \|V \Phi\|^2 - C_\epsilon \|\Phi\|^2$  and

$$\begin{aligned} \|(p^2 + V + H_f) \Phi\|^2 &\geq \|(p^2 + V) \Phi\|^2 + \|H_f \Phi\|^2 \\ &\geq (1 - \epsilon) \|p^2 \Phi\|^2 + \|V \Phi\|^2 - C_\epsilon \|\Phi\|^2 + \|H_f \Phi\|^2. \end{aligned}$$

Then (2.7) follows. ■

**Lemma 2.5** *Let  $V \in V_{\text{rel}} \cup V_{\text{conf}}$  and  $M \geq 0$ . Then  $\mathcal{H}_{\text{fin}}$  is a core of  $H$ .*

*Proof:* Suppose that  $M > 0$ . Let  $\Phi \in \mathcal{D}_2$ . Let  $V \in V_{\text{conf}}$ . Note that  $p^2 + V + H_f$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ . We see that there exists a sequence  $\{\Phi_n\}$ ,  $\Phi_n \in \mathcal{H}_{\text{fin}}$ , such that  $\Phi_n \rightarrow \Phi$ , and  $(p^2 + V + H_f) \Phi_n \rightarrow (p^2 + V + H_f) \Phi$  as  $n \rightarrow \infty$ . From (2.7) it follows that  $p^2 \Phi_n \rightarrow p^2 \Phi$ ,  $V \Phi_n \rightarrow V \Phi$  and  $H_f \Phi_n \rightarrow H_f \Phi$  as  $n \rightarrow \infty$ . Then we can also see that  $\{H \Phi_n\}$  is a Cauchy sequence by (2.2), and  $\lim_n H \Phi_n = H \Phi$  follows. Thus  $\mathcal{H}_{\text{fin}}$  is a core of  $H$ . Next we suppose that  $V \in V_{\text{rel}}$ . By the argument above it is seen that operator  $T_M + H_f$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ . By Proposition 1.4, we also



see that  $\|V\Phi\| \leq a\|(T_M + H_f)\Psi\| + b\|\Psi\|$  with some constant  $0 \leq a < 1$  and  $b \geq 0$ . The Kato-Rellich theorem yields that  $H$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ .

Suppose that  $M = 0$ . We emphasize the dependence on  $M$  by writing  $H_M$  for  $H$ . Since  $H_0 = H_M + (H_0 - H_M)$  and  $\|(H_0 - H_M)\Psi\| \leq M\|\Psi\|$ ,  $H_0$  is also essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$  by the fact that  $H_M$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$  and by the Kato-Rellich theorem.  $\blacksquare$

The key inequality to show the self-adjointness of  $H$  on  $\mathcal{D}$  is the following inequality.

**Lemma 2.6** *Let  $V \in V_{\text{conf}}$ . Let  $M_0 > 0$  be fixed and  $0 \leq M \leq M_0$ . Then for all  $\Psi \in D(H)$ ,*

$$\| |p|\Psi \|^2 + \|V\Psi\|^2 + \|H_f\Psi\|^2 \leq C\|(H + \mathbb{1})\Psi\|^2 \quad (2.8)$$

with some constant  $C$  independent of  $M$ .

*Proof:* Suppose that  $M = 0$ . In the case of  $M > 0$ , the proof is parallel with that of  $M = 0$ , but rather easier.

**(Step 0)** Let  $\Psi \in \mathcal{H}_{\text{fin}}$ . Let  $H_0 = |p - A| + H_f$ . We have

$$\begin{aligned} \|H\Psi\|^2 &= \|H_0\Psi\|^2 + \|V\Psi\|^2 + 2\text{Re}(H_0\Psi, V\Psi), \\ \|H_0\Psi\|^2 &= \| |p - A|\Psi \|^2 + \|H_f\Psi\|^2 + 2\text{Re}(|p - A|\Psi, H_f\Psi). \end{aligned}$$

Then

$$\|H\Psi\|^2 = \| |p - A|\Psi \|^2 + \|H_f\Psi\|^2 + 2\text{Re}(|p - A|\Psi, H_f\Psi) + \|V\Psi\|^2 + 2\text{Re}(H_0\Psi, V\Psi). \quad (2.9)$$

We estimate the three terms  $\| |p - A|\Psi \|^2$ ,  $\text{Re}(|p - A|\Psi, H_f\Psi)$  and  $\text{Re}(H_0\Psi, V\Psi)$  on the right-hand side of (2.9) from below.

**(Step 1)** We estimate  $\text{Re}(|p - A|\Psi, H_f\Psi)$ . Since the operator  $|p - A|$  is singular, we introduce an artificial positive mass  $m > 0$  and

$$T_m = \sqrt{(p - A)^2 + m^2}. \quad (2.10)$$

We fix  $m$  throughout. Note that  $|p - A| - T_m$  is bounded. Thus

$$|p - A| = T_m + (|p - A| - T_m) \quad (2.11)$$

can be regarded as a perturbation of  $T_m$ , and the perturbation  $|p - A| - T_m$  is bounded. We have  $\text{Re}(|p - A|\Psi, H_f\Psi) = \text{Re}(T_m\Psi, H_f\Psi) + \text{Re}((|p - A| - T_m)\Psi, H_f\Psi)$ . Since  $\Psi \in \mathcal{H}_{\text{fin}}$ ,  $H_f\Psi \in D(p^2) \cap D(H_f)$ . In particular  $H_f\Psi \in D(T_m)$  and then  $H_f\Psi \in D(T_m^{1/2})$ . Furthermore we show that

$$T_m^{1/2}\Psi \in D(H_f) \quad (2.12)$$

in Appendix D. So we can see that

$$\begin{aligned}
& \operatorname{Re}(|p - A|\Psi, H_f\Psi) \\
&= (T_m^{1/2}\Psi, H_f T_m^{1/2}\Psi) + \operatorname{Re}(T_m^{1/2}\Psi, [T_m^{1/2}, H_f]\Psi) + \operatorname{Re}((|p - A| - T_m)\Psi, H_f\Psi) \\
&\geq \operatorname{Re}(T_m^{1/2}\Psi, [T_m^{1/2}, H_f]\Psi) + \operatorname{Re}((|p - A| - T_m)\Psi, H_f\Psi).
\end{aligned}$$

We estimate  $((|p - A| - T_m)\Psi, H_f\Psi)$ . Since  $\|(|p - A| - T_m)\Psi\| \leq m\|\Psi\|$ , we see that for each  $\epsilon > 0$  there exists  $C_1 > 0$  such that

$$\operatorname{Re}((|p - A| - T_m)\Psi, H_f\Psi) \geq -\epsilon\|H_f\Psi\|^2 - C_1\|\Psi\|^2. \quad (2.13)$$

On the other hand we estimate  $\operatorname{Re}(T_m^{1/2}\Psi, [T_m^{1/2}, H_f]\Psi)$ . Let  $\epsilon > 0$  be given. Then there exists  $C_2 > 0$  such that

$$\begin{aligned}
\operatorname{Re}(T_m^{1/2}\Psi, [T_m^{1/2}, H_f]\Psi) &\geq -c\|T_m^{1/2}\Psi\| \|(H_f + \mathbb{1})\Psi\| \\
&\geq -\epsilon\||p - A|\Psi\|^2 - \epsilon\|H_f\Psi\|^2 - C_2\|\Psi\|^2.
\end{aligned} \quad (2.14)$$

The first inequality of (2.14) is derived from

$$\|[T_m^{1/2}, H_f]\Psi\| \leq c\|(H_f + \mathbb{1})^{1/2}\Psi\| \quad (2.15)$$

with some constant  $c > 0$ . This is shown in Appendix E. Hence we have

$$\operatorname{Re}(|p - A|\Psi, H_f\Psi) \geq -\epsilon\||p - A|\Psi\|^2 - 2\epsilon\|H_f\Psi\|^2 - (C_1 + C_2)\|\Psi\|^2. \quad (2.16)$$

**(Step 2)** We estimate  $\operatorname{Re}(H_0\Psi, V\Psi)$ . For each  $\epsilon > 0$  there exists  $C_3 > 0$  such that

$$\begin{aligned}
\operatorname{Re}(H_0\Psi, V\Psi) &= \operatorname{Re}((H_0 - T_m - H_f)\Psi, V\Psi) + \operatorname{Re}(T_m\Psi, V\Psi) + (H_f\Psi, V\Psi) \\
&\geq -\epsilon\|V\Psi\|^2 - C_3\|\Psi\|^2 + \operatorname{Re}(T_m\Psi, V\Psi).
\end{aligned}$$

We also see that

$$\operatorname{Re}(T_m\Psi, V\Psi) = (T_m^{1/2}\Psi, V T_m^{1/2}\Psi) + \operatorname{Re}(T_m^{1/2}\Psi, [T_m^{1/2}, V]\Psi) \geq \operatorname{Re}([T_m^{1/2}, V]\Psi, V\Psi).$$

Recall the integral representation

$$T_m^{1/2} = \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{w^{3/4}} (T_m^2 + w)^{-1} T_m^2 dw,$$

commutation relations

$$[(T_m^2 + w)^{-1} T_m^2, V] = (T_m^2 + w)^{-1} [T_m^2, V] - (T_m^2 + w)^{-1} [T_m^2, V] (T_m^2 + w)^{-1} T_m^2,$$

and facts

$$[T_m^2, V] = -2i \sum_{\mu=1}^d (p_\mu - A_\mu) V_\mu + \sum_{\mu=1}^d V_{\mu\mu},$$

where  $V_\mu = \partial_\mu V$  and  $V_{\mu\mu} = \partial_\mu^2 V$ . Then we have

$$\begin{aligned} |([T_m, V]\Psi, \Phi)| &= \left| \frac{1}{\sqrt{2\pi}} \int_0^\infty \frac{1}{w^{3/4}} [(T_m^2 + w)^{-1} T_m^2, V]\Psi, \Phi) dw \right| \\ &\leq \frac{\sqrt{2}}{\pi} \|\Psi\| \|\Phi\| \int_0^\infty \frac{dw}{w^{3/4}} \sum_{\mu=1}^d \left( \frac{2\|V_\mu\|_\infty}{\sqrt{w+m^2}} + \frac{\|V_{\mu\mu}\|_\infty}{w+m^2} \right). \end{aligned} \quad (2.17)$$

Thus for each  $\epsilon > 0$  there exists  $C_4 > 0$  such that

$$\operatorname{Re}(H_0\Psi, V\Psi) \geq -\epsilon\|V\Psi\|^2 - C_4\|\Psi\|^2. \quad (2.18)$$

**(Step 3)** We estimate  $\|p - A\|\Psi\|$ . Note that

$$\|p_\mu\Psi\|^2 = \|(p_\mu - A_\mu)\Psi\|^2 + 2\operatorname{Re}(A_\mu\Psi, (p_\mu - A_\mu)\Psi) + \|A_\mu\Psi\|^2.$$

For each  $\epsilon > 0$ , there exist  $C_5 > 0$  and  $C_6 > 0$  such that

$$\begin{aligned} |\operatorname{Re}(A\Psi, (p - A)\Psi)| &\leq \epsilon(\|p - A\|\Psi\|^2 + \|H_f\Psi\|^2) + C_5\|\Psi\|^2 \\ \|p\Psi\|^2 &\leq (1 + \epsilon)\|p - A\|\Psi\|^2 + \epsilon\|H_f\Psi\|^2 + C_6\|\Psi\|^2. \end{aligned}$$

Hence we have

$$\|p - A\|\Psi\|^2 \geq \frac{1}{1 + \epsilon} \|p\Psi\|^2 - \frac{\epsilon}{1 + \epsilon} \|H_f\Psi\|^2 - \frac{C_6}{1 + \epsilon} \|\Psi\|^2. \quad (2.19)$$

**(Step 4)** By (2.16), (2.18), (2.19) and (2.9), we can see (2.8) for  $\Psi \in \mathcal{H}_{\text{fin}}$ . Let  $\Psi \in D(H)$ . Since  $H$  is essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$ , by a limiting argument we can see (2.8) for  $\Psi \in D(H)$ .  $\blacksquare$

*Proof of Theorem 1.9:*

We emphasize the dependence on  $M$  by writing  $H_M$  for  $H$ . Let  $M > 0$ . Suppose that  $V \in V_{\text{conf}}$ . By Lemma 2.6,  $H_M$  is closed on  $\mathcal{D}$ . Then it implies that  $H_M$  is self-adjoint on  $\mathcal{D}$  since it is essentially self-adjoint on  $\mathcal{D}$ . Next suppose that  $V \in V_{\text{rel}}$ . Then  $T_M \dot{+} H_f$  is self-adjoint on  $\mathcal{D}$ . Since  $V$  is also relatively bounded with respect to  $T_M \dot{+} H_f$  with a relative bound strictly smaller than one. Thus  $H$  is self-adjoint on  $\mathcal{D}$ .

Let  $M = 0$ . By  $H_0 = H_M + (H_0 - H_M)$  and  $\|(H_0 - H_M)\Psi\| \leq M\|\Psi\|$ ,  $H_0$  is self-adjoint on  $\mathcal{D}$  and essentially self-adjoint on  $\mathcal{H}_{\text{fin}}$  by the Kato-Rellich theorem.  $\blacksquare$

### 3 Translation invariant case

The momentum operator in  $\mathcal{F}$  is defined by the second quantization of the multiplication by  $k_\mu$ . I.e.,  $P_{f\mu} = \sum_{r=1}^{d-1} \int k_\mu a^{\dagger r}(k) a^r(k) dk$ ,  $\mu = 1, \dots, d$ . Let  $P_{\text{tot}\mu} = p_\mu \otimes \mathbb{1} + \mathbb{1} \otimes P_{f\mu}$ ,  $\mu = 1, \dots, d$ , be the total momentum operator, and we set  $P_{\text{tot}} = (P_{\text{tot}1}, \dots, P_{\text{tot}d})$ . Let  $V = 0$ . Then we can see that  $[H, P_{\text{tot}\mu}] = 0$  and hence  $H$  can be decomposed with respect to the spectrum of  $P_{\text{tot}\mu}$ . Thus  $H \cong \int_{\mathbb{R}^d}^{\oplus} H_P dP$ , where  $H_P$  is called the fiber Hamiltonian with the total momentum  $P \in \mathbb{R}^d$ .

We can see the explicit form of the fiber Hamiltonian. Let

$$L(P) = (P - P_f - A(0))^2 + M^2. \quad (3.1)$$

**Proposition 3.1** [Hir07, Theorem 2.3 (2), Lemma 3.11] *Let  $P \in \mathbb{R}^d$ . Then  $L(P)$  is essentially self-adjoint on  $\mathcal{C}_0 = D(P_f^2) \cap D(H_f)$ .*

Set

$$\bar{L}(P) = \overline{L(P)}|_{\mathcal{C}_0}. \quad (3.2)$$

**Definition 3.2** *Let  $P \in \mathbb{R}^d$ . We define  $H(P)$  by*

$$H(P) = \sqrt{\bar{L}(P)} \dot{+} H_f. \quad (3.3)$$

**Lemma 3.3** *It follows that*

$$T_M + H_f \cong \int_{\mathbb{R}^d}^{\oplus} H(P) dP. \quad (3.4)$$

*Proof:* We define the unitary operator  $U$  on  $\mathcal{H}$  by  $(UF)(\cdot) \in \mathcal{H}$  for  $F(\cdot) \in \mathcal{H}$  by

$$(UF)(P) = (2\pi)^{-d/2} \int_{\mathbb{R}^d} e^{iP \cdot x} e^{-iP_f \cdot x} F(x) dx. \quad (3.5)$$

It is shown that

$$U^{-1} \left( \int_{\mathbb{R}^d}^{\oplus} \bar{L}(P) dP \right) U = (p - A)^2 \quad (3.6)$$

in [Hir07, Theorem 2.3]. Actually it is shown that

$$(F, T_M^2 G) = \int_{\mathbb{R}^d} dP ((UF)(P), \bar{L}(P)(UG)(P))_{\mathcal{F}} \quad (3.7)$$

for  $F, G \in \mathcal{H}_{\text{fin}}$ . From (3.6) we see that  $U^{-1} \left( \int_{\mathbb{R}^d}^{\oplus} e^{-t\bar{L}(P)} dP \right) U = e^{-tT^2}$  for all  $t \geq 0$  by [RS78, Theorem XIII 85 (c)]. Let  $F \in \mathcal{H}_{\text{fin}}$ . By the formula

$$T_M^\alpha = C_\alpha \int_0^\infty (\mathbb{1} - e^{-\lambda T_M^2}) \frac{d\lambda}{\lambda^{1+\alpha/2}}, \quad (3.8)$$

we can see that

$$(F, T_M F) = C_1 \int_0^\infty \frac{d\lambda}{\lambda^{3/2}} \int_{\mathbb{R}^d} dP ((UF)(P), (\mathbb{1} - e^{-\lambda \bar{L}(P)})(UF)(P)).$$

By Fubini's theorem we have

$$(F, T_M F) = C_1 \int_{\mathbb{R}^d} dP \int \frac{d\lambda}{\lambda^{3/2}} ((UF)(P), (\mathbb{1} - e^{-\lambda \bar{L}(P)})(UF)(P)). \quad (3.9)$$

Note that  $(UF)(P) \in \mathcal{F}_{\text{fin}}$  for each  $P \in \mathbb{R}^d$ . Hence  $(UF)(P) \in D(\bar{L}(P)) \subset D(\sqrt{\bar{L}(P)})$ , which implies that

$$(F, T_M F) = \int_{\mathbb{R}^d} dP \left( (UF)(P), \sqrt{\bar{L}(P)}(UF)(P) \right). \quad (3.10)$$

By the polarization identity and (3.10) we have

$$(F, T_M G) = \int_{\mathbb{R}^d} dP \left( (UF)(P), \sqrt{\bar{L}(P)}(UG)(P) \right).$$

Furthermore we see that

$$(F, (T_M + H_f) G) = \int_{\mathbb{R}^d} dP ((UF)(P), H(P)(UG)(P)),$$

which implies that

$$T_M \dot{+} H_f = U^{-1} \left( \int_{\mathbb{R}^d}^{\oplus} H(P) dP \right) U \quad (3.11)$$

on  $\mathcal{H}_{\text{fin}}$ . Since  $\mathcal{H}_{\text{fin}}$  is a core of the left hand side of (3.11),

$$T_M \dot{+} H_f \cong \int_{\mathbb{R}^d}^{\oplus} H(P) dP \quad (3.12)$$

holds true as self-adjoint operators. Note that  $T_M \dot{+} H_f = T_M + H_f$  on  $D(|p|) \cap D(H_f)$  and  $T_M + H_f$  is self-adjoint on  $D(|p|) \cap D(H_f)$ . Then the lemma follows.  $\blacksquare$

Let  $\mathcal{C} = D(|P_f|) \cap D(H_f)$ . Note that  $D(|P - P_f|) = D(|P_f|)$  for all  $P \in \mathbb{R}^d$ . The essential self-adjointness of  $H(P)$  is established in [Hir14].

**Proposition 3.4** [Hir14, Corollary 7.2] *Let  $M > 0$ . Then  $H(P)$  is essentially self-adjoint on  $\mathcal{C}$ .*

The main result in this section is as follows.

**Theorem 3.5** *Let  $M \geq 0$ . Then  $H(P)$  is self-adjoint on  $\mathcal{C}$  and essentially self-adjoint on  $\mathcal{F}_{\text{fin}}$ .*

*Proof:* The proof is parallel with that of  $H$ . We show the outline of the proof. It can be seen that there exists a constant  $C > 0$  such that for arbitrary  $\Psi \in \mathcal{F}_{\text{fin}}$ ,

$$\|\sqrt{(P - P_f - A(0))^2 + M^2}\Psi\| \leq C(\| |P - P_f| \Psi \| + \| H_f^{1/2} \Psi \| + \| \Psi \|).$$

Then we can derive that

$$\|H(P)\Psi\| \leq C(\| |P - P_f| \Psi \| + \| H_f \Psi \| + \| \Psi \|) \quad (3.13)$$

for  $\Psi \in \mathcal{F}_{\text{fin}}$ . In a similar manner to Lemma 2.2 from (3.13) we can see that  $\mathcal{C}_1 = \mathcal{C} \cap \mathcal{F}_{\text{fin}}$  is a core of  $H(P)$  for  $M > 0$ . Since  $|P - P_f|^2$  and  $H_f$  are strongly commutative and positive, it is trivial to see that

$$\|(|P - P_f|^2 + H_f)\Psi\|^2 \geq \| |P - P_f|^2 \Psi \|^2 + \| H_f \Psi \|^2. \quad (3.14)$$

Since  $\mathcal{F}_{\text{fin}}$  is a core of  $|P - P_f|^2 + H_f$ , in a similar manner to Lemma 2.5 we can see that  $\mathcal{F}_{\text{fin}}$  is also a core of  $H(P)$  by (3.14). The key inequality to show the self-adjointness of  $H(P)$  is

$$\| |P - P_f| \Psi \|^2 + \| H_f \Psi \|^2 \leq C \|(H(P) + \mathbb{1})\Psi\|^2 \quad (3.15)$$

with some  $C > 0$  for  $\Psi \in \mathcal{F}_{\text{fin}}$ . This is shown by using the inequality

$$\|[T_m(P)^{1/2}, H_f]\Psi\| \leq c\|(H_f + \mathbb{1})^{1/2}\Psi\|, \quad (3.16)$$

where  $T_m(P) = \sqrt{(P - P_f - A(0))^2 + m^2}$ . (3.16) is proven in Appendix F. Thus by (3.15) in a similar manner to the proof of Theorem 1.9 we can see that  $H(P)|_{\mathcal{C}}$  is closed. Then  $H(P)$  is self-adjoint on  $\mathcal{C}$  for  $M \geq 0$ .  $\blacksquare$

## A Stochastic preliminary

In this appendix we review functional integral representations of the semigroup generated by semi-relativistic Pauli-Fierz model. This is established in [Hir14, Theorem 3.13]. These representations play an important roles to estimate some commutation relations in this paper.

## A.1 Semi-relativistic Pauli-Fierz model

Let  $(B_t)_{t \geq 0}$  be the  $d$ -dimensional Brownian motion defined on a Wiener space with Wiener measure  $P^x$  starting from  $x$ . Let  $(T_t)$  be the subordinator on a probability space with a probability measure  $\nu$  such that  $\mathbb{E}_\nu[e^{-uT_t}] = e^{-t(\sqrt{2u+M^2}-M)}$ . We denote the expectation with respect to the measure  $P^x \otimes \nu$  by  $\mathbb{E}_{P^x \otimes \nu}^x[\cdot \cdot \cdot]$ . Let  $a = (a_1(x), \dots, a_d(x))$  be electromagnetic fields. Then the semi-relativistic Schrödinger operator is defined by  $h = \sqrt{(p-a)^2 + M^2} - M + V$ . Then the Feynman-Kac formula [LHB11, Chapter 3] yields the path integral representation of  $e^{-th}$  by

$$(f, e^{-th}g) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P^x \otimes \nu}^x \left[ e^{-\int_0^t V(B_{T_s}) ds} e^{-i \int_0^t a(B_s) dB_s} \overline{f(B_{T_0})} g(B_{T_t}) \right]. \quad (\text{A.1})$$

On the other hand the semi-relativistic Pauli-Fierz model is defined by the minimal coupling of  $h + H_f$  with a quantized radiation field  $A$ :

$$H = T_M \dot{+} V \dot{+} H_f.$$

We can give the functional integral representation of  $e^{-tH}$  in [Hir14, Theorem 3.13]. Let

$$q(F, G) = \frac{1}{2} \sum_{\mu, \nu=1}^d (\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu)$$

be the quadratic form on  $\oplus^d L^2(\mathbb{R}^d)$ , where  $\delta_{\mu\nu}^\perp(k) = \delta_{\mu\nu} - k_\mu k_\nu / |k|^2$  denotes the transversal delta function. Let  $\mathcal{A}(F)$  be a Gaussian random variables on a probability space  $(Q, \Sigma, \mu)$ , which is indexed by  $F = (F_1, \dots, F_d) \in \oplus^d L^2(\mathbb{R}^d)$ . The mean of  $\mathcal{A}(F)$  is zero and the covariance is given by  $\mathbb{E}[\mathcal{A}(F)\mathcal{A}(G)] = q(F, G)$ . Furthermore we introduce the Euclidean version of  $\mathcal{A}$ . Let

$$q_E(F, G) = \frac{1}{2} \sum_{\mu, \nu=1}^d (\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu) \quad (\text{A.2})$$

be the quadratic form on  $\oplus^d L^2(\mathbb{R}^{d+1})$ . On the right-hand side of (A.2), we note that  $(\hat{F}_\mu, \delta_{\mu\nu}^\perp \hat{G}_\nu) = \int_{\mathbb{R} \times \mathbb{R}^d} \overline{\hat{F}_\mu(k_0, k)} \delta_{\mu\nu}^\perp(k) \hat{G}_\nu(k_0, k) dk_0 dk$  and  $\delta_{\mu\nu}^\perp(k)$  is independent of  $k_0$ . Let  $\mathcal{A}_E(F)$  be a Gaussian random variables on a probability space  $(Q_E, \Sigma_E, \mu_E)$ , which is indexed by  $F \in \oplus^d L^2(\mathbb{R}^{d+1})$ . The mean of  $\mathcal{A}_E(F)$  is zero and the covariance is given by  $\mathbb{E}[\mathcal{A}_E(F)\mathcal{A}_E(G)] = q_E(F, G)$ . Let us identify  $\mathcal{H}$  with  $L^2(\mathbb{R}^d; \mathcal{F})$ . Thus  $\Phi \in \mathcal{H}$  can be an  $\mathcal{F}$ -valued  $L^2$ -function on  $\mathbb{R}^d$ ,  $\mathbb{R}^d \ni x \mapsto \Phi(x) \in \mathcal{F}$ . It is well known that there exists the family of isometries  $J_t : L^2(Q) \rightarrow L^2(Q_E)$  ( $t \in \mathbb{R}$ ) and  $j_t : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^{d+1})$  ( $t \in \mathbb{R}$ ) such that  $J_t^* J_s = e^{-|t-s|H_f}$  and  $j_t^* j_s = e^{-|t-s|\omega(-i\nabla)}$ .

**Proposition A.1** *Let  $F, G \in \mathcal{H}$ . Then*

$$(F, e^{-tH}G) = e^{-tM} \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x \left[ e^{-\int_0^t V(B_{T_s}) ds} (J_0 F(B_{T_0}), e^{-i\mathcal{A}_E(I[0,t])} J_t G(B_{T_t}))_{L^2(Q_E)} \right]. \quad (\text{A.3})$$

Here  $I[0, t] = \oplus_{i=1}^d \int_0^{T_t} j_{T^*s} \tilde{\varphi}(\cdot - B_s) dB_s^i$  is defined by the limit of  $\oplus^d L^2(\mathbb{R}^{d+1})$ -valued stochastic integrals of  $\tilde{\varphi} = (\hat{\varphi}/\sqrt{\omega})$ , and  $T_s^* = \inf\{t; T_t = s\}$ .

*Proof:* See [Hir14, Theorem 3.13 and Remark 3.8]. ■

Furthermore let

$$K = \frac{1}{2}(p - A)^2 \quad (\text{A.4})$$

be the kinetic term of the Pauli-Fierz model  $K + V + H_f$ . The Feynman-Kac formula of  $e^{-tK}$  is also established as follows.

**Proposition A.2** *Let  $F, G \in \mathcal{H}$ . Then it follows that*

$$(F, e^{-tK}G) = \int_{\mathbb{R}^d} dx \mathbb{E}_P^x \left[ e^{-\int_0^t V(B_{T_s}) ds} (F(B_0), e^{-i\mathcal{A}(K[0,t])} G(B_t))_{L^2(Q)} \right], \quad (\text{A.5})$$

where  $K[0, t] = \oplus_{i=1}^d \int_0^t \tilde{\varphi}(\cdot - B_s) dB_s^i$  is a  $\oplus^d L^2(\mathbb{R}^d)$ -valued stochastic integral.

*Proof:* See [Hir00, (4.20), Theorem 4.8] and [LHB11, (7.3.18)]. ■

## A.2 Semi-relativistic Pauli-Fierz model with a fixed total momentum

Let  $H(P) = \sqrt{(P - P_f - A(0))^2 + M^2} + H_f$  be the semi-relativistic Pauli-Fierz model with total momentum  $P \in \mathbb{R}^d$ . The rigorous definition of  $H(P)$  is given by (3.3). The Feynman-Kac formula of  $e^{-tH(P)}$  is also established.

**Proposition A.3** *Let  $F, G \in L^2(Q)$ . Then*

$$(F, e^{-tH(P)}G) = e^{-tM} \mathbb{E}_P^0 \left[ (J_0 F(B_{T_0}), e^{-i\mathcal{A}_E(I[0,t])} e^{i(P - P_f) \cdot B_{T_t}} J_t G(B_{T_t}))_{L^2(Q_E)} \right]. \quad (\text{A.6})$$

*Proof:* This is proven by a minor modification of [Hir07, Theorem 3.3]. ■

Furthermore the kinetic term of the Pauli-Fierz model with total momentum  $P \in \mathbb{R}^d$  is given by

$$K(P) = \frac{1}{2}(P - P_f - A(0))^2, \quad P \in \mathbb{R}^d.$$

The Feynman-Kac formula of  $e^{-tK(P)}$  is also established as follows.

**Proposition A.4** *Let  $F, G \in L^2(Q)$ . Then*

$$(F, e^{-tK(P)}G) = \mathbb{E}_P^0 \left[ (F(B_0), e^{-i\mathcal{A}(K[0,t])} e^{i(P - P_f) \cdot B_t} G(B_t))_{L^2(Q)} \right]. \quad (\text{A.7})$$

*Proof:* This is also proven by a minor modification of [Hir07, Theorem 3.3]. ■



## B Proof of Lemma 1.3

*Proof of Lemma 1.3:*

It is shown that  $e^{-tK}$  leaves  $D(p^2) \cap C^\infty(N)$  invariant in [LHB11, Lemma 7.53]. See also [Hir00, Theorem 2.6]. It is enough to show that  $e^{-tK}D(H_f) \subset D(H_f)$ . By the Feynman-Kac formula we have

$$\begin{aligned} (H_f F, e^{-tK} G) &= \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(H_f F(B_0), e^{-i\mathcal{A}(K[0,t])} G(B_t))] \\ &= (F, e^{-tK} H_f G) + \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\mathcal{A}(K[0,t])}] G(B_t))]. \end{aligned}$$

We can estimate as  $[H_f, e^{-i\mathcal{A}(K[0,t])}] = e^{-i\mathcal{A}(K[0,t])}(\Pi(K[0,t]) + \xi)$ , where  $\Pi(K[0,t]) = [H_f, \mathcal{A}(K[0,t])]$  and  $\xi = q(K[0,t], K[0,t])$ . Thus we see that

$$\left| \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\mathcal{A}(K[0,t])}] G(B_t))] \right| \leq C(t + \sqrt{t}) \|F\| \| (H_f + \mathbb{1})^{1/2} G \|. \quad (\text{B.1})$$

Here we used that  $\|\Pi(K[0,t])\Psi\| \leq C(\|K[0,t]\| + \|K[0,t]/\sqrt{\omega}\|) \|(H_f + \mathbb{1})^{1/2}\Psi\|$  and BDG-type inequality ([Hir00, Theorem 4.6] and [LHB11, Lemma 7.21]):

$$\mathbb{E}_P^0[\xi^2] \leq t^2 C, \quad (\text{B.2})$$

$$\mathbb{E}_P^0[(\|K[0,t]\| + \|K[0,t]/\sqrt{\omega}\|)^2] \leq Ct. \quad (\text{B.3})$$

Then we have

$$|(H_f F, e^{-tK} G)| \leq C(t + \sqrt{t}) \|F\| \| (H_f + \mathbb{1})^{1/2} G \| + \|F\| \|H_f G\|,$$

and the desired results follow. ■

## C Proof of Proposition 1.8

**Lemma C.1** *Let  $V \in V_{\text{conf}}$ . Then  $e^{-tH}$  leaves  $D(V)$  invariant, i.e.,  $e^{-tH}D(V) \subset D(V)$ .*

*Proof:* Let  $F, G \in D(V)$ . We define  $Q_{[0,t]}$  by  $Q_{[0,t]} = e^{-tM} e^{-\int_0^t V(B_s) ds} J_0^* e^{-i\mathcal{A}_E(I[0,t])} J_t : \mathcal{H} \rightarrow \mathcal{H}$ . Then we have

$$(VF, e^{-tH}G) = \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x[(V(B_{T_0})F(B_{T_0}), Q_{[0,t]}G(B_{T_t}))].$$

Hence we see that

$$(VF, e^{-tH}G) = (F, e^{-tH}VG) + \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x [(F(B_{T_0}), Q_{[0,t]}(V(B_0) - V(B_{T_t}))G(B_{T_t}))]$$

and, by the Taylor expansion  $V(x) - V(B_{T_t} + x) = \sum_{\mu} (\partial_{\mu} V(\xi)) B_{T_t}^{\mu}$  with some  $\xi \in \mathbb{R}^d$ , we can estimate as

$$\left| \int_{\mathbb{R}^d} dx \mathbb{E}_{P \times \nu}^x [(F(B_{T_0}), Q_{[0,t]}(V(B_0) - V(B_{T_t}))G(B_{T_t}))] \right| \leq \|F\| \| |x|G \| \sup_x \sqrt{\sum_{\mu} |\partial_{\mu} V(x)|^2}.$$

Here we used the fact that  $G \in D(|x|)$ . Then we have

$$|(VF, e^{-tH}G)| \leq C \|F\| (\| |x|G \| + \|VG\|)$$

with some constant  $C > 0$ . Then  $e^{-tH}G \in D(V)$  follows.  $\blacksquare$

*Proof of Proposition 1.8:*

Suppose that  $V$  satisfies (2) of Assumption 1.7. It is shown in [Hir14, Lemmas 4.3 and 4.4] that  $D(H) \subset \cap_{\mu} D(p_{\mu}) \cap D(H_f)$  and  $e^{-tH}$  leaves  $\cap_{\mu} D(p_{\mu}) \cap D(H_f)$  invariant, which implies that  $e^{-tH}$  leaves  $D(|p|) \cap D(H_f)$  invariant. Combining this with Lemma C.1 we see that  $D(H) \subset \mathcal{D}$  and  $e^{-tH}$  leaves  $\mathcal{D}$  invariant. Then  $\mathcal{D}$  is a core of  $H$  by [RS75, Theorem X.49].  $\blacksquare$

## D Proof of (2.12)

Note that  $T_m^{1/2} = (2K + m^2)^{1/4}$ , where  $K$  is given by (A.4). We have

$$(2K + m^2)^{\alpha/2} = C_{\alpha} \int_0^{\infty} (\mathbb{1} - e^{-\lambda(2K+m^2)}) \frac{d\lambda}{\lambda^{1+\alpha/2}} \quad (\text{D.1})$$

for  $0 \leq \alpha < 2$  with some constant  $C_{\alpha}$ . From this formula we have the lemma below:

**Lemma D.1** *There exists  $C > 0$  such that*

$$(F, T_m^{1/2}G) = C \int_0^{\infty} \left\{ (F, G) - e^{-\lambda m^2/2} \int_{\mathbb{R}^d} dx \mathbb{E}_P^x [(F(B_0), e^{-i\mathcal{A}(K[0,\lambda])} G(B_{\lambda}))] \right\} \frac{d\lambda}{\lambda^{5/4}}.$$

*Proof:* This can be derived from Proposition A.2, (D.1) and changing the variable.  $\blacksquare$

*Proof of (2.12):*

Let  $F \in D(H_f)$  and  $G \in \mathcal{H}_{\text{fin}}$ . Thus  $H_f G \in \mathcal{H}_{\text{fin}}$ . By (D.1) we have

$$(H_f F, T_m^{1/2}G) = C \int_0^{\infty} \left\{ (H_f F, G) - e^{-\lambda m^2/2} \int_{\mathbb{R}^d} dx \mathbb{E}_P^x [(H_f F(B_0), e^{-i\mathcal{A}(K[0,\lambda])} G(B_{\lambda}))] \right\} \frac{d\lambda}{\lambda^{5/4}}.$$

Then we have

$$\begin{aligned} & (H_f F, T_m^{1/2} G) - (F, T_m^{1/2} H_f G) \\ &= -C \int_0^\infty \frac{e^{-\lambda m^2/2}}{\lambda^{5/4}} d\lambda \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] G(B_\lambda))]. \end{aligned} \quad (D.2)$$

We have

$$[H_f, e^{-i\mathcal{A}(K[0,\lambda])}] = e^{-i\mathcal{A}(K[0,\lambda])}(\Pi(K[0,\lambda]) + \xi),$$

where  $\Pi(K[0,\lambda]) = [H_f, \mathcal{A}(K[0,\lambda])]$  and  $\xi = q(K[0,\lambda], K[0,\lambda])$ . Thus we see that in a similar manner to (B.1), (B.2) and (B.3),

$$\begin{aligned} & \left| \int_0^\infty \left\{ \int_{\mathbb{R}^d} dx \mathbb{E}_P^x[(F(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] G(B_\lambda))] \right\} \frac{e^{-\lambda m^2/2} d\lambda}{\lambda^{5/4}} \right| \\ & \leq C \int_0^\infty \frac{\sqrt{\lambda} + \lambda}{\lambda^{5/4}} e^{-\lambda m^2/2} d\lambda \|F\| \| (H_f + \mathbb{1})^{1/2} G \|. \end{aligned} \quad (D.3)$$

Then we see that  $|(H_f F, T_m^{1/2} G)| \leq C \|F\| \| (H_f + \mathbb{1})^{1/2} G \|$  with some constant  $C > 0$ . Hence  $T_m^{1/2} G \in D(H_f)$  follows.  $\blacksquare$

## E Proof of (2.15)

*Proof of (2.15):*

The proof of (2.15) is similar to that of (2.12). Let  $G \in \mathcal{H}_{\text{fin}}$ . By (D.2) and (D.3), it follows that  $|(F, [H_f, T_m^{1/2}] G)| \leq C \|F\| \| (\mathbb{1} + H_f)^{1/2} G \|$ . This implies (2.15).  $\blacksquare$

## F Proof of (3.16)

*Proof of (3.16):*

The idea of the proof of (3.16) is similar to (2.12) and (2.15). We have

$$(\Phi, [H_f, T_m(P)^{1/2}] \Psi) = (H_f \Phi, T_m(P)^{1/2} \Psi) - (\Phi, T_m(P)^{1/2} H_f \Psi)$$

The Feynman-Kac formula yields that

$$(\Phi, [H_f, T_m(P)^{1/2}] \Psi) = \int_0^\infty \frac{e^{-m^2 \lambda/2} d\lambda}{\lambda^{5/4}} \mathbb{E}_P^0[e^{iP \cdot B_\lambda} (\Phi(B_0), [H_f, e^{-i\mathcal{A}(K[0,\lambda])}] e^{-iP_f \cdot B_\lambda} \Psi(B_\lambda))].$$

Since  $[H_f, e^{-i\mathcal{A}(K[0,\lambda])}] = e^{-i\mathcal{A}(K[0,\lambda])}(\Pi(K[0,\lambda]) + \xi)$ . Then in a similar manner to (D.3) we can derive the desired results.  $\blacksquare$

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